Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

20 May 2026

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

Instructions to students

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

7 pages

2226 – 5022

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 26]

This question asks you to explore a game of chance.

A game of chance involves attempting to win sweets from a basket through a series of rounds. The rules of the game are:

- The player starts the game with at least one sweet.
- At the start of each round the player spins a spinner.
- If the spinner shows "win", the player wins a sweet from the basket; otherwise, they must return one sweet to the basket.
- The game is played until either all the sweets are won, or the player has no sweets remaining.

Initially, consider the case where there is a total of three sweets in the game.

Fiona starts the game with one sweet and there are two sweets in the basket to win.

The probability of obtaining a "win" in any round is $\frac{1}{4}$, independent of any other spin.

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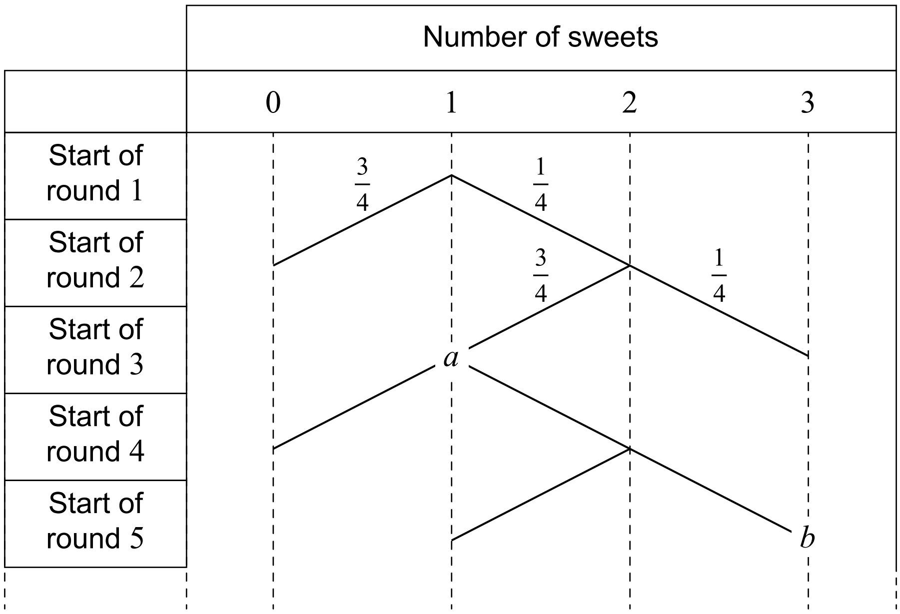
(Question 1 continued)

The following diagram shows the possible number of sweets that Fiona has at the start of each round, and the possible outcomes of the initial rounds.



If Fiona wins round 1 she will have exactly two sweets at the start of round 2.

If she loses round 1, she will have no sweets remaining at the start of round 2 and the game ends.

The probability of Fiona having exactly one sweet at the start of round 3 is a, and the probability of her having three sweets at the start of round 5 is b.

(a) (i) Show that $a = \frac{3}{16}$. [1]

(ii) Find the value of b. [2]

At the start of round 5, Fiona has exactly one sweet.

(b) Write down the probability that she will still have one sweet after two more rounds. [2]

(This question continues on the following page)

(Question 1 continued)

Now consider the case where there is a total of w sweets in the game.

Let R_n denote the probability that Fiona will eventually have all w sweets, when she starts a round with n sweets.

If Fiona finishes a round with no sweets, $n = 0$, the probability that she will eventually have all the sweets, R_0, is zero and the game ends.

For $n > 0$, if Fiona wins the next round, she will have $n + 1$ sweets and the probability of her eventually having all w sweets is now R_{n+1}.

Similarly, if Fiona loses the next round, she will have $n - 1$ sweets and the probability of her eventually having all w sweets is now R_{n-1}.

The probability that the spinner shows "win" is now p, where $0 < p < 1$, and the probability that it shows "not win" is q, where $q = 1 - p$ and $p \neq q$.

At the start of each round, R_n is given by

$$R_n = qR_{n-1} + pR_{n+1}, \text{ where } n > 0.$$

Consider the case where $w = 4$ and $p = \dfrac{1}{4}$.

(c) Write down an expression for R_2 in terms of R_1 and R_3. [1]

(d) It is known that when Fiona starts a round with exactly one sweet, the probability that she will win the game is $R_1 = 0.025$. When she starts a round with exactly three sweets, the probability that she will win the game is $R_3 = 0.325$.

Find the value of R_2. [2]

(This question continues on the following page)

(Question 1 continued)

For the following parts of this question, consider the case where there is a total of w sweets in the game and the probability that the spinner shows "win" is p.

(e) By expressing R_n as $(p+q)R_n$, show that $R_{n+1} - R_n = \dfrac{q}{p}\left(R_n - R_{n-1}\right)$. [2]

(f) (i) Hence, find an expression for $R_2 - R_1$ in terms of p, q and R_1. [1]

(ii) Show that $R_3 - R_2 = \left(\dfrac{q}{p}\right)^2 R_1$. [2]

(iii) Deduce an expression for $R_n - R_{n-1}$ in terms of n, p, q and R_1. [1]

(g) (i) By considering $(R_2 - R_1) + (R_3 - R_2) + (R_4 - R_3) + \ldots + (R_n - R_{n-1})$, or otherwise, show that

$$R_n = R_1\left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \ldots + \left(\frac{q}{p}\right)^{n-1}\right).$$ [4]

(ii) Hence, given that $R_1 = \dfrac{1-\dfrac{q}{p}}{1-\left(\dfrac{q}{p}\right)^w}$, show that $R_n = \dfrac{1-\left(\dfrac{q}{p}\right)^n}{1-\left(\dfrac{q}{p}\right)^w}$. [4]

There is now a large number of sweets in the game.

Fiona starts a game with m sweets, where $p = 0.495$.

Once she has obtained 10 additional sweets, she will have all the sweets and the game ends.

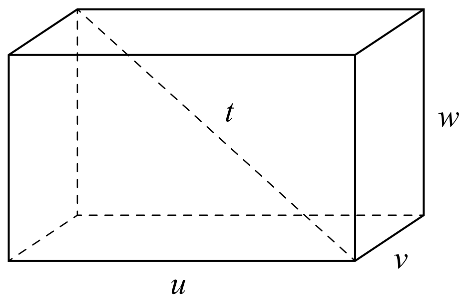
(h) Determine the minimum value of m for Fiona to have at least an 80% chance of winning 10 additional sweets. [4]

2. [Maximum marks: 29]

This question asks you to explore the relationship between complex numbers and a cuboid whose dimensions and interior diagonal are all integer lengths. This cuboid is called a Pythagorean box.

A Pythagorean box, denoted as $\mathrm{PB}(u, v, w, t)$, is a cuboid with dimensions $u \times v \times w$ and an interior diagonal of length t, where $u, v, w, t \in \mathbb{Z}^+$. This is shown in the following diagram.

diagram not to scale



(a) For the case where $u = 9$, $v = 2$ and $w = 6$, find the value of t. [2]

The complex numbers z_1 and z_2 are used to generate a Pythagorean box, $\mathrm{PB}(u, v, w, t)$, if

$u = 2\,\mathrm{Re}(z_1 z_2^*)$, $v = 2\,\mathrm{Im}(z_1 z_2^*)$, $w = |z_1|^2 - |z_2|^2$ and $t = |z_1|^2 + |z_2|^2$, where $u, v, w, t \in \mathbb{Z}^+$.

Conversely, for any Pythagorean box, $\mathrm{PB}(u, v, w, t)$, there exist two complex numbers, z_1 and z_2, which satisfy u, v, w and t as defined above.

Consider the two complex numbers $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}$, where $a, b, c, d \in \mathbb{Z}^+$.

(b) (i) Find an expression for $z_1 z_2^*$ in terms of a, b, c and d. [2]

(ii) Hence, find an expression for v in terms of a, b, c and d. [1]

(c) Consider the case where $z_1 = 3 + 7\mathrm{i}$ and $z_2 = c + 6\mathrm{i}$ are used to generate a Pythagorean box.

(i) Show that $v = 14c - 36$. [1]

(ii) Show that the only possible values of c are 3 and 4. [4]

(iii) For $c = 4$, find the values of u, v, w and t. [5]

(This question continues on the following page)

(Question 2 continued)

There are three Pythagorean boxes with an interior diagonal of length 23.

(d) Given that $23 = 1^2 + 2^2 + 3^2 + 3^2$, determine the dimensions of **one** of these Pythagorean boxes. [6]

The dimensions and interior diagonal of the Pythagorean box, $\mathrm{PB}(u, v, w, t)$, can be used to form a complex number, z, such that $z = \dfrac{u + v\mathrm{i}}{t - w}$.

Consider the Pythagorean box $\mathrm{PB}(26, 38, 43, 63)$.

(e) Show that, for this Pythagorean box, $z = \dfrac{13 + 19\mathrm{i}}{10}$. [1]

The complex number z can also be written as $\dfrac{z_1}{z_2}$, where z_1 and z_2 generate the Pythagorean box $\mathrm{PB}(26, 38, 43, 63)$.

It is given that $\dfrac{z_1}{z_2}$ can be expressed as $\dfrac{z_1 z_2^{\,*}}{|z_2|^2}$.

(f) (i) By considering $t - w$, show that $c^2 + d^2 = 10$. [2]

(ii) Hence, given that $c > d$, find z_1 and z_2. [5]